Roundness measurement using limacons

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Most roundness measuring systems which find automatically the 'reference circles' to a profile have used, for convenience, a limacon figure as an approximation to a circle. However this figure has important geometrical properties relevant to normal roundness measurement and can be regarded as the basis of an analytical system rather than an instrumental convenience. This paper examines some of the implications of the limacon method under practical conditions by comparison with circular references. It includes a comparison of roundness measurements taken using limacon references according to minimum zone, minimum circumscribing, maximum inscribing and least squares criteria

Nomenclature

(a,b)	Cartesian position of eccentricity
(E,ϕ)	Polar position of eccentricity
L	Suppressed radius
R	'Component' Radius
S	Nominal chart radius
$\gamma = \frac{E}{R}$	Eccentricity ratio
ϵ	Measured displacement
M	Instrument magnification
W	Annular width of polar chart

The basic questions which a metrologist might ask about an object concern its size, its position and its shape. With nominally circular parts the radius, centre and deviations of shape (out-of-roundness) are the variables of interest, but the meaning of 'radius' and 'centre' may not be clear when the shape varies irregularly. So conventional definitions must be agreed to allow comparison of measurements from different sources. This is done by using reference figures, perfect geometrical shapes which are related to the imperfect form by some 'best-fit' criterion (least squares, for example). The radius and centre position are then related to the reference figure parameters and the shape imperfections are measured in terms of deviations, usually radial, from the reference figure.

In national standards the defined reference figures are true circles. For reasons of computational convenience however, it is common to find that the circle is somewhat approximated in practice. Usually such approximation involves use of the figure known as the limacon. In particular, the great majority of, if not all, automatic roundness assessing systems use limacons. It is well known that the shape difference between a circle and a limacon is similar to the shape distortion which is introduced into the profile by the instrumental technique and so compensates for this distortion. The limacon approximation has become widely, if not formally, accepted on pragmatic grounds.

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Here by comparing the geometries of limacons and circles in the light of the philosophy of roundness measurement, it is argued that the limacon has a more fundamental role than it has previously been afforded. This is not only of theoretical interest; there is considerable potential for easing the practical constraints currently employed in roundness measurement. A greater tolerance to the presence of eccentricity in the measurement is given: this may increase the efficiency of use of current instruments and could indeed influence the design of future instruments.

Roundness measurement

Most measurements of nearly circular parts use a precision spindle to create a relative rotational motion between the part and a displacement transducer mounted radially with respect to the axis of rotation. The signal from this represents a combination of the out-of-roundness of the part and the variation of radial distance of the surface from the axis caused by relative eccentricity between them. Since the very small size of the out-of-roundness compared to the radius causes a severe problem of range:resolution in the transducer, only the variation of the signal is measured to high precision and subsequently processed, the absolute radius of the part being lost or at least preserved only to a lower precision. This 'radius suppression' is illustrated in Fig 1 which shows schematically the measurement of a circular part. The transducer output, ϵ, indicates the separation of the component and a perfect, centred circle with radius equal to the suppression. (The polar chart which is used to display the magnified roundness errors is an example of radius suppressed information.)

It is convenient to describe this situation by the effects in three co-ordinate systems. Firstly there are hypothetical component co-ordinates in which the points

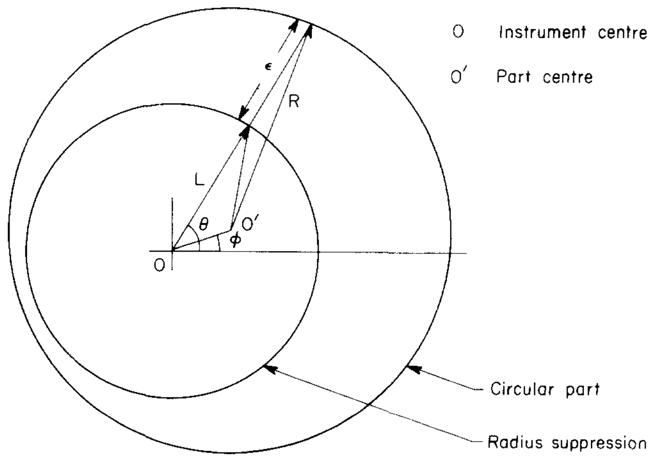

Fig 1 Roundness measurement with radius suppression

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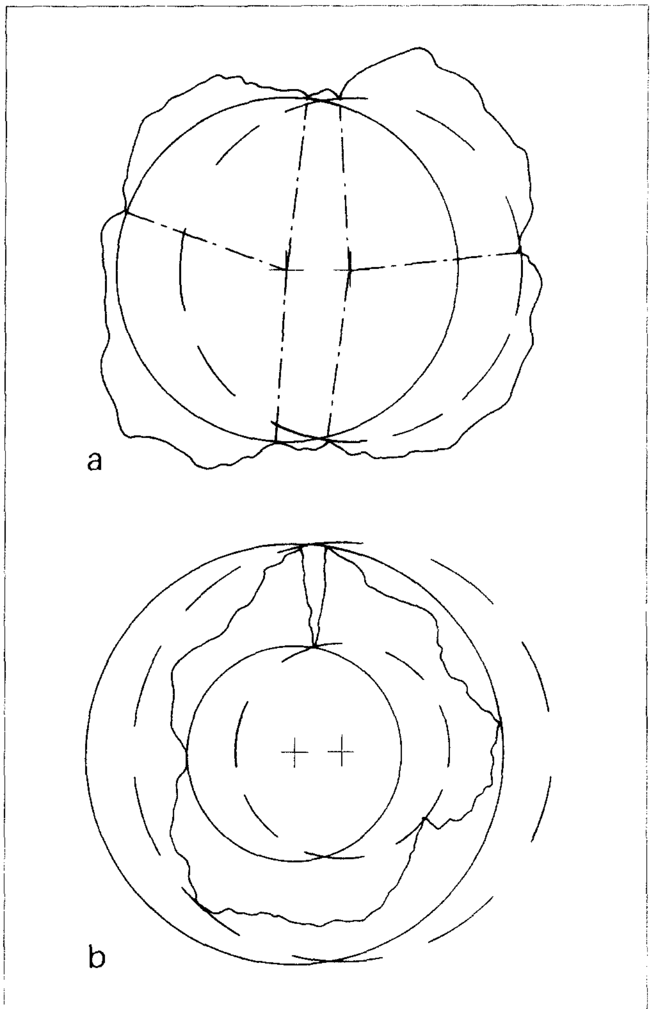

Fig 2 Alternative reference circles for maximum inscribing and minimum zone criteria

of the surface are defined only with respect to each other. Then to measure the surface it is expressed relative to instrument co-ordinates which locate it in space and inevitably include in its description effects due to the presence of eccentricity. Finally, because of practical considerations, the surface is expressed in a set of chart co-ordinates related to instrument co-ordinates by a radius suppression transformation. The use of a reference figure is an attempt to get back from instrument to component co-ordinates. In practice it is necessary to translate the desired reference into chart co-ordinates.

Circular references

If it is assumed that the component is ideally circular, the most logical reference figure would be a circle. This circle must then be related to the surface profile being measured. The national standards suggest four ways for doing this[1] :

- Least Squares Circle (LSC)
- Minimum Radial Zone Circles (MZC)
- Minimum Radius Circumscribing Circle (MCC)
- Maximum Radius Inscribing Circle (MIC)

Since only radius-suppressed data is available, the standards relate to fitting these circles in chart co-ordinates. Using the notation from Fig 1, an eccentric circle in instrument co-ordinates is:

$$\rho(\theta) = E\cos(\theta - \phi) + (R^2 - E^2\sin^2(\theta - \phi))^{1/2}$$

$$= E\cos(\theta - \phi) + R - \frac{E^2}{2R}\sin^2(\theta - \phi) \quad \ldots\ldots (1)$$

on expanding by the binomial series. The chart co-ordinate equivalent to this will be :

$$\epsilon(\theta) = E\cos(\theta - \phi) + R - L - \frac{E^2}{2R}\sin^2(\theta - \phi) \ldots\ldots (2)$$

This form is normally magnified and plotted on an arbitrary radius, but, in any case, it does not plot as a circle. The degree of 'distortion' depends, for a given radius suppression, upon the amount of eccentricity present. Thus to effectively fit reference circles in instrument co-ordinates requires that non-circular shapes be fitted on the chart. Alternatively, fitting circular references on the chart is an approximation to what is really wanted! The distortion and magnification of the profile on the chart also tend to emphasise another problem of circle fitting: with both MZC and MIC certain profile shapes can have more than one local minimum or maximum. Fig 2 shows examples of these.

The circle is an ideal figure for evaluating graphs 'by hand' using either a transparent template marked with concentric circles or a pair of compasses. It is surprisingly inconvenient to handle mathematically as would be required in automatic instruments. It is noteworthy that the only analytical method (for LSC) given in British Standards[1] uses approximations. There appears, then, to be a case for defining a non-circular reference for use in chart co-ordinates providing that it can be related to circles in instrument co-ordinates.

Approximate references

Before considering a particular alternative to circular references, the properties required of an ideal reference will be examined.

It should closely model the form displayed by an eccentric (or centred) circle in both instrument and chart co-ordinates. To facilitate translation between co-ordinate systems it should depend on the degree of radius suppression only very simply, if at all. On any profile a theoretically unique reference should exist for all fitting criteria. The ability to express its shape in a mathematically continuous form is useful. Definitive algorithms for its calculation according to all the fitting criteria should exist and be reasonably easy to implement in practice. It must also be possible to use the method 'by hand' directly on a graph. It would be preferable if, for at least a large set of restricted conditions, measurements with the reference figure could be related easily to those performed with circles in order to maintain continuity of records.

If a reference system could meet all these conditions there would be a case for adopting it as a standard: the current version of standards cannot satisfy all of them. All the reference fitting criteria which have been proposed represent, in mathematical terms, problems in optimisation. One of the few classes of problem to which optimisation theory has a complete solution is that in which the functions have only a linear dependence on their parameters. Further, the theory guarantees that there is just one global optimum in such problems. So, a reference with linear parameters would be advantageous.

Limacon references

The normal approximation given for solving the least squares circle [1] is merely to truncate the series (2) to give

$$\epsilon(\theta) \approx E\cos(\theta - \phi) + (R-L) \quad (3)$$

$$= a\cos\theta + b\sin\theta + R-L \quad (4)$$

It was recognised[2] that this describes a specific geometric figure, namely a limacon and that its shape represents more closely the radius suppressed form of a circle than does a circle in chart co-ordinates. Consider the display on a polar chart when a radius-suppressed circle is magnified by M and plotted relative to an arbitrary radius, S, giving:

$$\rho(\theta) = M\epsilon + S$$

$$= ME\cos(\theta - \phi) + M(R-L) + S - \frac{ME^2}{2R}\sin^2(\theta - \phi) \ldots \quad (5)$$

The corresponding limacon would be:

$$r_L = M\,(E\cos(\theta - \phi) + R-L) + S \quad (6)$$

and a circle with the same parameters:

$$r_C = ME\cos(\theta - \phi) + M(R-L) + S - \frac{M^2E^2}{2(M(R-L) + S)}\sin^2(\theta - \phi) \ldots \quad (7)$$

The second order term is much larger in equation (7) than in the true form (equation (5)). Ignoring second order terms altogether as in the limacon (equation (6)) is a better representation providing that, approximately:

$$2(M\,(R-L) + S) < MR \quad (8)$$

A typical value for $M(R-L) + S$ might be 50 mm, so the limacon will be the preferable model of polar distortion at magnifications greater than 100 on 1 mm radius components, and only 10 on 10 mm radius components. Virtually all roundness measurements will therefore benefit from the use of limacons although as E becomes small the error in both shapes decreases and becomes negligble. At zero eccentricity, circle and limacon are, of course, identical.

When expressed in the form of equation (4), the limacon satisfies the linearity requirements discussed in the previous section; indeed, it is for that reason that it is so amenable to least squares analysis. Recent developments[3] have, by exploiting this, led to simple geometric 'exchange-algorithms' for solving minimum zone, minimum radius circumscribing and maximum radius inscribing limacons, where the term 'radius' is taken to mean the constant term of equation (3).

As the position of the co-ordinate system origin is unaffected by the radius suppression transformation, the parameter linearity ensures that the geometrical properties of a limacon are preserved under radius suppression or its inverse. So a limacon in instrument co-ordinates translates to a limacon, albeit with different parameter values, in chart co-ordinates. Furthermore the relationship between a profile and a reference limacon is unaltered by radius suppression. If identical conditions of radius suppression and magnification, or their reversal, are applied to a profile and limacon, then, for instance, the minimum radius circumscribing figure remains minimum circumscribing and the least squares limacon retains that property[4]. This is not the case with circular references.

Although the true reference shape in chart co-ordinates is usually better represented by a limacon than a circle, in instrument co-ordinates the circle is correct and the limacon an approximation to it. The acceptability of limacon references depends on the quality of this approximation. The power series expansion of equation (1) is in terms of γ^2 ($\gamma<1$) and so will die away quite rapidly. The 'error' between limacon and circle will be dominated by the second term giving:

$$\text{Error} \approx \frac{E^2}{2R}\sin^2(\theta - \phi) = \frac{\gamma^2 R}{4}(1 - \cos 2(\theta - \phi)) \quad (9)$$

In practice the eccentricity ratio rarely exceeds 0.01, so the radial variation between the limacon and circle is at most a fraction of a percent of the total signal caused by eccentricity.

As the expansion of expression (1) contains only even powers of sin $(\theta-\phi)$, the equivalent Fourier series will contain no odd harmonics other than the limacon term. Thus theoretically the least squares limacon will identify exactly the centre of a perfect circle. From equation (9) the radius term of the least squares limacon will underestimate the radius of a perfect circle by about $\gamma^2 R/4$. The MCC radius is estimated by the equivalent limacon to better than $\pm\gamma^2 R/4$. The centre of this circle is also well estimated by the limacon providing that certain geometrical conditions are met, as is usually the case in practice[4].

The magnitude of the errors just discussed may be brought into perspective by considering typical instrumentation. The profile is constrained to fit within the polar chart annulus at any magnification, M. The system resolution will be say, 0.001 W/M (0.1 % full scale) which corresponds to the commonly used 10-bit digitisation. The largest possible eccentricity for a circle is then $W/2M$, and this will cause error terms of about:

$$\frac{\gamma E}{4} = \frac{\gamma W}{8M}$$

So for $\gamma<0.008$, a quite large value, the errors will be less than the system resolution!

The limacon references are seen to be good approximations to circles in instrument co-ordinates under normal conditions and further have, in γ, a simple indicator of their quality.

Limacons and circles in practice

It is normal practice to limit the eccentricity ratio on the chart by a criterion derived from a maximum allowable deviation (0.25 mm in American Standards[5]) of a radius suppressed circle from a true circle, since circular references are to be used on the chart[6]. Subject to the condition in equation (8), it would be expected that this criterion could be relaxed when using limacon references.

In Fig 3, as an illustration, the eccentricity, E_{max}, which causes a maximum deviation on the chart of 0.25 mm between the profile of a true circle and its limacon and circular references is plotted against its radius. It is assumed that the magnification is 1000, the radius suppression complete ($R=L$) and the chart radius, S, is 50 mm. Generally the limacon is more tolerant of eccentricity than the circle, since the eccentricity ratio in instrument co-ordinates is much smaller than in chart co-ordinates. As the radius approaches S/M, there is effectively no radius suppression and the circle becomes true even at large eccentricity.

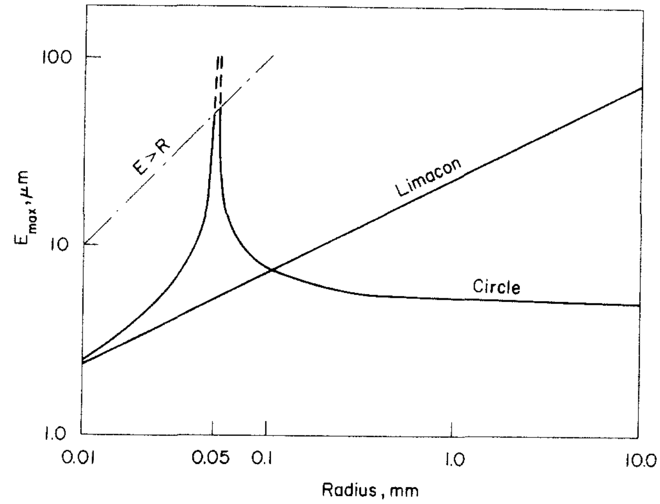

Fig 3 Allowable eccentricity, E_{max} *for given maximum error between reference and circular part on the polar chart*

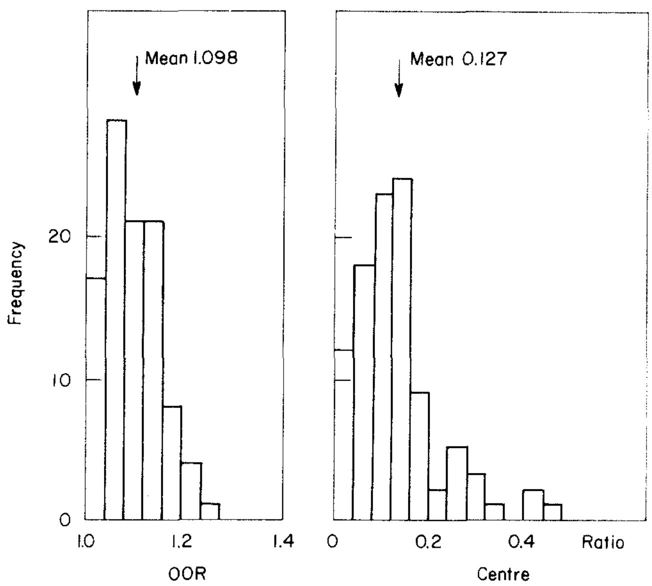

Fig 4 Comparison of least squares limacons to minimum zone limacons on typical parts

A practical consequence of the greater tolerance to eccentricity given by the limacon is the 'de-skilling' of roundness measurement. This term should be taken in a wide sense. Not only could it allow relatively unskilled operators to perform their own roundness checks but it could also, by reducing the time needed for refinement of centring, reduce the measurement cycle time. It may be possible to use considerably cheaper fixturing without loss of accuracy and make the introduction of automatic handling more plausible. When working at very high magnification the requirements for fitting circles can become almost prohibitive, whereas limacons still allow relatively simple setting-up procedures.

The main disadvantage of limacon references compared to circles is the difficulty of using them by hand on a polar graph. Limacon 'compasses' can be made but are inconvenient to use and since the shape of a limacon varies with its eccentricity template methods are hardly practicable[7]. Limacons are best suited to machine computation. This problem is not serious since providing the current guidelines for maximum allowable eccentricity are followed, a circle may be used instead of a limacon with errors no greater than those to be expected of any graphical method. Further, a consistent measurement philosophy has been achieved: firstly the eccentricity ratio in instrument co-ordinates is limited so that the limacon well approximates a circle there, then if a circle is to be used on the chart the eccentricity ratio there is held small so that the circle adequately approximates the equivalent limacon. Current centring guidelines ignore the first instrument co-ordinate check. In most cases this is quite safe: when the radius suppression is large the second criterion dominates the first.

The general behaviour of limacon reference systems has been investigated experimentally. A set of 100 profiles were measured from randomly selected components which might be found in any precision engineering laboratory. Most common engineering materials and forming processes were included in the set and about one third were holes. Component diameters varied between 5 mm and 100 mm, , the instrument magnifications ranging between 500 and 20 000. The majority of the parts had out-of-roundness values between 1 µm and 10 µm which is, perhaps, the most common range of use of roundness instruments. Each component was required to be set-up on the instrument at a magnification reasonable for measuring its out-of-roundness from a graph but without any particular care being taken to minimise eccentricity. By storing the digitised profiles (512 equiangular points) on magnetic disc it was possible to compare different analysis techniques on exactly the same profiles.

In comparing various reference systems, there is a

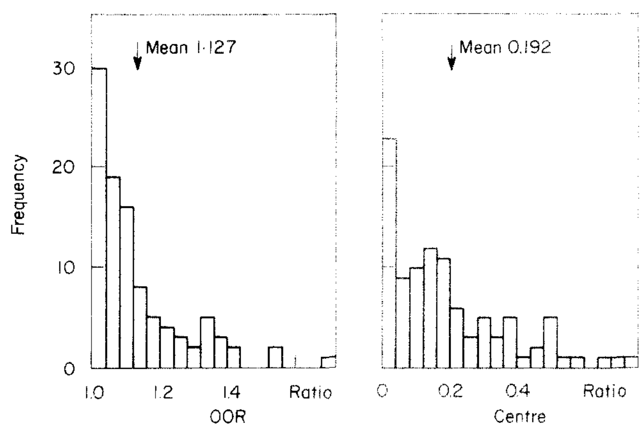

Fig 5 Comparison of minimum circumscribing limacons to minimum zone limacons on typical parts

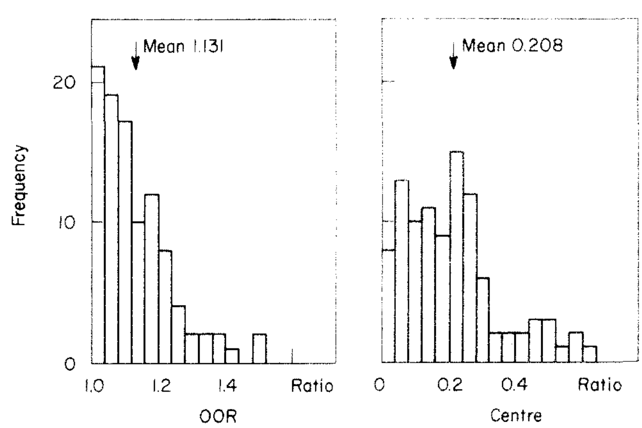

Fig 6 Comparison of maximum inscribing limacons to minimum zone limacons on typical parts

problem in relating all the information to a common basis. One way of doing this is to express out-of-roundness, measured as peak-to-valley, in terms of the ratio of that measured by the particular system to the minimum zone. Variations in centre are expressed in terms of the distance of the centre from the minimum zone centre normalised to the minimum zone width[5]. This approach, being a reasonable solution to the normalisation difficulty, is adopted because it allows direct comparison with earlier work. Figs 4–6 show the distribution of out-of-roundness values and centre separations obtained with the least squares, minimum circumscribing and maximum inscribing limacons relative to the minimum zone limacons. The general form of these histograms agrees quite closely with the results given in reference 5 which are for circles fitted on the chart by hand. The least squares and minimum zone limacons do, it appears, tend to have separate identities but are rarely much different. On the other hand there is a quite high probability of the minimum circumscribing limacon being very close to minimum zone although more widely differing values occur than with least squares. The maximum inscribing condition seems less tied to minimum zone, particularly for centre. It is interesting to note that there was no particular tendency for inscribing and zone conditions to agree on holes and circumscribing and zone to agree on shafts.

On this set of measurements, the eccentricity ratio in instrument co-ordinates never exceeded 0.003, while in chart co-ordinates 61 were above 0.1 and 24 above 0.2. Reason's recommendations for centring to 7 % or 15 % eccentricity ratio on the chart were violated on 78 and 42 occasions respectively. Thus although the conditions for limacon measurements were good, considerably more care would have been needed for working with circles on the chart. This was emphasised by an interactive circle fitting program which was used with the same data. No arithmetically significant difference was detected between the parameters of the minimum circumscribing limacon and circle in instrument co-ordinates. The same comparison in chart co-ordinates showed 40 instances when differences were found. These differences were often not serious when compared to the accuracy obtained by graphical methods but serve to emphasise the added confidence obtainable by using limacons.

Concluding remarks

By formally studying the motivation for using reference figures in the component–instrument–chart co-ordinate frames, the properties needed of reference figures can be identified. The limacon, notably because of its linearity, satisfies such requirements. It is more precise than a circular reference under nearly all practical measurement conditions. The circle is more easily used graphically but it is regarded as an approximation to a limacon on the polar chart. The eccentricity ratio provides a consistent test in instrument and chart co-ordinates for such approximations.

The limacon provides various advantages over the circle, but maintains compatibility with current practice in roundness measurement.

It might be argued that since under many practical conditions the differences between limacons and circles are small, the differentiation is of little importance. Such a view overlooks the effect that the formal use of limacons might have in fields as diverse as legally enforceable standards, future developments in instrumentation and the cost effectiveness of automatic or semi-automatic measuring stations.

Acknowledgement

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